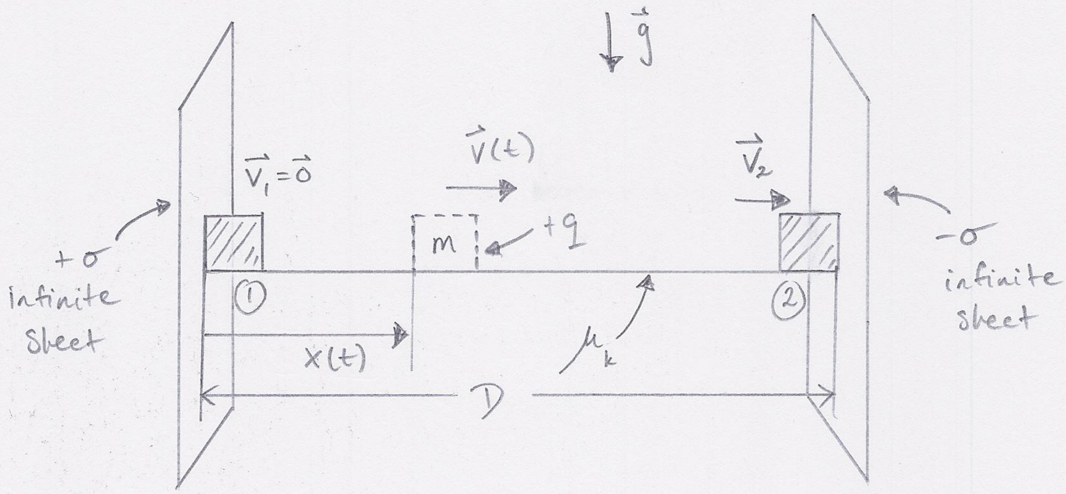


A block of mass "m" and charge "+q" is placed at rest on a surface with friction. The block is placed infinitely close to a positively charged infinite sheet without actually touching the sheet.

Find the equivalent resistance of this circuit.

$$\Delta V = iR$$

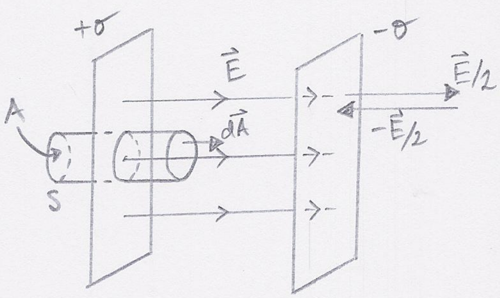
$$R = \frac{\Delta V}{i} \quad \text{——— (1)}$$

In order to find the resistance, we need to find the potential difference "ΔV" between the two terminals of the source that incentivises current "i" to flow.

Therefore "ΔV" will be the potential difference between the two infinite plates.

$$\Delta V = -\int_a^b \vec{E}\cdot d\vec{x} \quad \text{——— (2)}$$

Let's find $\vec{E}$:



Note: $\vec{E}$-Field outside plates cancels out!

$$\oint_S \vec{E}\cdot d\vec{A} = \frac{Q_{enc}}{\varepsilon_0}$$

$$EA = \frac{\sigma A}{\varepsilon_0}$$

$$E = \frac{\sigma}{\varepsilon_0} \quad \text{——— (3)}$$

③ $\longrightarrow$ ②

$$\Delta V = -\int_0^D \frac{\sigma}{\varepsilon_0}\,dx$$

$$\Delta V = -\frac{\sigma x}{\varepsilon_0}\Big|_0^D$$

$$\Delta V = -\frac{\sigma D}{\varepsilon_0}$$

Although our block is losing potential energy, we are really only interested in the magnitude of "ΔV" in order to find "R".

$\therefore$ let $\Delta V = \dfrac{\sigma D}{\varepsilon_0}$ ——————— (2')

(2') $\longrightarrow$ ①

$R = \dfrac{(\sigma D/\varepsilon_0)}{i}$ ——————— (1')

Now we must find the current "i," which is the rate of change of charge with respect to time through a given cross sectional area.

Intuitively, we would think that current is linearly related to the velocity of the charged block.

That is $i \propto v(t)$

Let's look at the relationship between current and energy dissipated.
Note that the energy dissipated in this circuit due to resistance is equal to the work done on the block by friction.

That is:

$$E_{diss} = \int P\,dt$$

$$P = i\,\Delta V$$

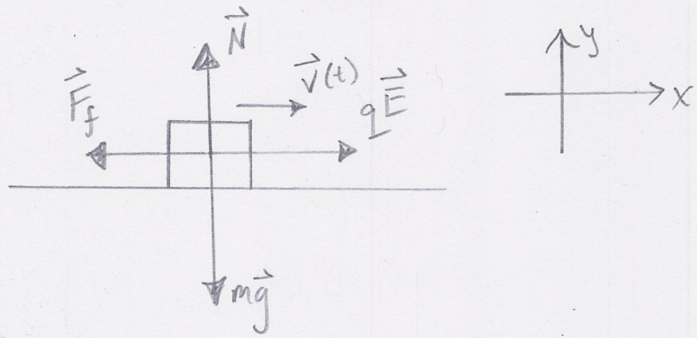
$$\Delta V = iR$$

$$\therefore \; E_{diss} = \int i^2 R\,dt = W_f$$

$$i^2 R = \frac{d}{dt}(W_f) \qquad (4)$$

Now let's find the work done by friction using the definition of work:

$$W_f = \int_a^b \vec{F}_f \cdot d\vec{x} \qquad (5)$$



$\vec{F} = m\vec{a}$:

x

$$qE - F_f = ma \text{ ——— } (6)$$

y

$$N - mg = 0 \text{ ——— } (7)$$

$$F_f = \mu_k N \text{ ——— } (8)$$

↳ We will assume the block breaks static friction for convenience.

$$(7)\quad N - mg = 0$$

$$N = mg \longrightarrow (8)$$

$$F_f = \mu_k mg \text{ ——— } (8')$$

$$(8') \longrightarrow (5)$$

$$W_f = \int_0^{x(t)} \mu_k mg \, dx$$

$$W_f = \mu_k mg \, x \Big|_0^{x(t)}$$

$$W_f = \mu_k mg \, x(t) \text{ ——— } (5')$$

$(5') \longrightarrow (4)$

$$i^2R = \frac{d}{dt}\left(\mu_k mg\, x(t)\right)$$

$$i^2R = \mu_k mg \frac{dx(t)}{dt}$$

$$i^2R = \mu_k mg\, v(t)$$

$$\therefore \quad i = \sqrt{\frac{\mu_k mg\, v(t)}{R}} \longrightarrow (1')$$

$$R = \frac{(\sigma D/\varepsilon_0)}{\sqrt{\frac{\mu_k mg\, v(t)}{R}}}$$

$$R^2 = \frac{(\sigma D/\varepsilon_0)^2}{\left(\frac{\mu_k mg\, v(t)}{R}\right)}$$

$$R^{\cancel{2}} = \frac{\sigma^2 D^2}{\varepsilon_0^2} \cdot \frac{\cancel{R}}{\mu_k mg\, v(t)}$$

$$\therefore \quad R = \frac{\sigma^2 D^2}{\mu_k mg\, \varepsilon_0^2\, v(t)} \longrightarrow (1'')$$

We can find $v(t)$ by referring back to our force equations:

$$qE - F_f = ma \longrightarrow 6$$

$$F_f = \mu_k mg \longrightarrow 8'$$

$$3 \ \& \ 8' \longrightarrow 6$$

$$q\left(\frac{\sigma}{\varepsilon_0}\right) - (\mu_k mg) = ma$$

$$\therefore \quad a(t) = \frac{q\sigma}{m\varepsilon_0} - \mu_k g$$

$$v(t) = \left[\frac{q\sigma}{m\varepsilon_0} - \mu_k g\right]t + \cancelto{0}{v_0}$$

$$x(t) = \frac{1}{2}\left[\frac{q\sigma}{m\varepsilon_0} - \mu_k g\right]t^2 + \cancelto{0}{x_0}$$

$$\therefore \ v(t) = \left(\frac{q\sigma}{m\varepsilon_0} - \mu_k g\right)t \longrightarrow 1''$$

$$R = \frac{\sigma^2 D^2}{\mu_k mg\varepsilon_0^2\left[\left(\frac{q\sigma}{m\varepsilon_0} - \mu_k g\right)t\right]}$$

$$R = \frac{\sigma^2 D^2}{\mu_k m g\, \varepsilon_0^2 \left[\left(\frac{q\sigma - \mu_k g m \varepsilon_0}{m \varepsilon_0} \right) t \right]}$$

$$R = \frac{\sigma^2 D^2 \cancel{m} \cancel{\varepsilon_0}}{\mu_k \cancel{m} g\, \varepsilon_0^{\cancel{2}} \left(q\sigma - \mu_k m g \varepsilon_0 \right) t}$$

$$\therefore \quad \boxed{R(t) = \frac{\sigma^2 D^2}{\mu_k g \varepsilon_0 \left[q\sigma - \mu_k m g \varepsilon_0 \right]} \cdot \frac{1}{t}}$$

Let's check units:

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{enc}}{\varepsilon_0}$$

$$\therefore \left[\frac{N}{C} \right] [m]^2 = \frac{[C]}{[\varepsilon_0]}$$

$$\therefore [\varepsilon_0] = \left[\frac{C^2}{N\,m^2} \right]$$

$$[R(t)] = \frac{\left[\frac{C}{m^2} \right]^2 [m]^2}{[\cancel{\mu_k}]^1 \left[\frac{m}{s^2} \right] \left[\frac{C^2}{Nm^2} \right] \left[[C] \left[\frac{C}{m^2} \right] - [\cancel{\mu_k}]^1 \cdot [kg] \left[\frac{m}{s^2} \right] \left[\frac{C^2}{Nm^2} \right] \right] [s]}$$

$$= \frac{\left[\frac{C^2 m^2}{m^4} \right]}{\left[\frac{m}{s^2} \right] \left[\frac{C^2}{Nm^2} \right] \left[\left[\frac{C^2}{m^2} \right] - [\cancel{N}] \left[\frac{C^2}{\cancel{N} m^2} \right] \right] [s]}$$

$$= \frac{\cancel{\left[\frac{C^2}{m^2}\right]}}{\left[\frac{m\,C^2}{s^2 m^2}\right]\left[\frac{1}{N}\right]\cancel{\left[\frac{C^2}{m^2}\right]}\left[s\right]}$$

$$= \frac{[N]}{\left[\frac{m\,C^2}{m^2 s}\right]}$$

$$= \frac{\left[\frac{kg\,m}{s^2}\right]}{\left[\frac{C^2}{m\,s}\right]}$$

$$= \left[\frac{kg\,m}{s^2}\right]\left[\frac{m\,s}{C^2}\right]$$

$$= \frac{\left[\frac{kg\,m^2}{s^2}\right]}{[C]} \cdot \frac{[s]}{[C]}$$

$$= \left[J/C\right] \cdot \frac{1}{[C/s]}$$

$$= \frac{[V]}{[A]}$$

$$\therefore\ [R(t)] = [\Omega] \checkmark$$

Note:

1''
$$R = \frac{\sigma^2 D^2}{\mu_k m g \varepsilon_0^2 v(t)}$$

$$R = \frac{\left(\frac{\sigma^2 D^2}{\varepsilon_0^2}\right)}{\mu_k m g v(t)}$$

2'
$$\frac{\sigma D}{\varepsilon_0} = \Delta V$$

$$\therefore R = \frac{\Delta V^2}{\mu_k m g v(t)}$$

$$R = \frac{\Delta V}{\left(\frac{\mu_k m g v(t)}{\Delta V}\right)}$$

which implies

$$i(t) = \frac{\mu_k m g}{\Delta V} v(t)$$

Notice μ_k, m, g, and ΔV are all constants. This supports our prediction that

$$i \propto v(t).$$

where $\frac{\mu_k m g}{\Delta V}$ is the proportionality constant!

Let's check units:

$$[i] = \frac{\cancel{[\mu_k]}^{1} [kg][m/s^2]}{[V]} [m/s]$$

$$= \frac{[N]}{[J/C]} \cdot [m/s]$$

$$= \frac{[N \cdot m]\ [C]}{[J]\ [s]}$$

$$= \frac{\cancel{[J]}}{\cancel{[J]}} \left[\frac{C}{s}\right]$$

$$[i] = [A] \checkmark$$